Nonintegrability of three-degree-of-freedom Birkhoff normal forms of resonance degree two

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1 Introduction

In this paper, we study integrability of three-degree-of-Hamiltonian systems in Birkhoff normal form. Let H be a real analytic function of $z = (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ and assume it has the following power series expansion:

$$H(z) = \sum_{j=2}^{\infty} H^j(z), \quad H^2 = \sum_{k=1}^{3} \frac{\omega_j}{2}(x_j^2 + y_j^2), \tag{1}$$

where H^j represents homogeneous terms of degree j and $\omega_1, \omega_2, \omega_3 > 0$ are constants. The Hamiltonian H is said to be in *Birkhoff normal form* if

$$\{H, H^2\} := \sum_{j=1}^{3} \left(\frac{\partial H}{\partial x_j}\frac{\partial H^2}{\partial y_j} - \frac{\partial H^2}{\partial x_j}\frac{\partial H}{\partial y_j} \right) = 0.$$

It is well known that there exists a formal symplectic transformation $z = \phi(\zeta)$ such that $H \circ \phi$ is in Birkhoff normal form. In this case, $H \circ \phi$ is called a Birkhoff normal form of H and ϕ is called a *normalization* of H. If the equilibrium is *non-resonant*, i.e., $\omega_1, \omega_2, \omega_3$ are rationally linear independent, then the Hamiltonian vector field of the normal form $H \circ \phi$ is linear. On the other hand, the equilibrium is resonant, that is, $\omega_1, \omega_2, \omega_3$ are rationally linear dependent, the Hamiltonian vector field of $H \circ \phi$ may not be linear. Hence the set

$$R := \{\gamma_k \in \mathbb{Z}^3 \mid \sum_{k=1}^{m} \gamma_k \omega_k = 0\},$$

which is called the *resonance set* of H, plays an important role in the normal form theory. The number $\gamma := \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}} R$ is the *resonance degree*.

The central problem is to give a sufficient condition that the formal normalization ϕ is analytic in the neighborhood of the equilibrium. We fucus on the results related with integrability. An m-degree-of-freedom Hamiltonian system is called analytically (resp.

meromorphically) integrable if there exist m analytic (resp. meromorphic) functions $H_1 = H, H_2, \ldots, H_m$ such that $\{H_j, H_k\} = 0$ for $j, k = 1, \ldots m$. Ito [4, 5] showed that if a Hamiltonian with the resonance degree $\gamma \leq 1$ is analytically integrable, then there exists an analytic normalization of H. Zung[11] proved without any assumptions about resonance degrees that if a Hamiltonian is analytically integrable, then it has an analytic normalization. This means that an integrable system is transformed to an integrable normal form in the analytic framework. However, a Hamiltonian in Birkhoff normal form may not be integrable: there exists an analytically nonintegrable Birkhoff normal form with resonance degree $\gamma \geq 2$, while Birkhoff normal forms with $\gamma \leq 1$ are always integrable (see [5]).

The purpose of the paper is to study integrability of (1) in Birkhoff normal form with resonance degree 2. We assume $(\omega_1, \omega_2, \omega_3) = (1, 2, \omega), \omega = 1, 2, 3$ or 4 and that (1) is in Birkhoff normal form and a cubic polynomial. Integrability and dynamics of these systems have been studied (see [1, 3, 10]). Our Hamiltonian systems for $\omega = 1, 2, 3$ and 4 can be analytically transformed into the following Hamiltonian systems, respectively:

$$H = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_2(p_3^2 - q_3^2) + 2p_3q_3q_2], \tag{ω-1}$$

$$H = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_3(p_1^2 - q_1^2) + 2p_1q_1q_3], \tag{ω-2}$$

$$H = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_3(p_1p_2 - q_1q_2) + q_3(q_1p_2 + p_1q_2)], \tag{ω-3}$$

$$H = a[p_2(p_1^2 - q_1^2) + 2p_1q_1q_2] + b[p_3(p_2^2 - q_2^2) + 2p_2q_2q_3], \tag{ω-4}$$

where $a, b \in \mathbb{R}$ are parameters, by some time-dependent transformation

$$\begin{aligned} x_j &= \cos(\omega_j t + \delta_j)q_j + \sin(\omega_j t + \delta_j)p_j, \\ y_j &= -\sin(\omega_j t + \delta_j)q_j + \cos(\omega_j t + \delta_j)p_j, \end{aligned}$$

where δ_j are constants. Without loss of generality, we can assume that $a, b \geq 0$ and that $a \geq b \geq 0$ for (ω-1).

A Hamiltonian in Birkhoff normal form has a first integral H^2. Moreover, the Hamiltonian (ω-2) has another first integral

$$\frac{a^2 + b^2}{2}(p_1^2 + q_1^2) + (aq_2 + bq_3)^2 + (ap_2 + bp_3)^2.$$

Hence we need only to study the case of $\omega = 1, 3, 4$. It is apparent that $(q_3^2 + p_3^2)/2$ is a first integral when $b = 0$. Moreover, the Hamiltonian (ω-1) has a first integral

$$q_1p_3 - p_1q_3$$

when $a = b$ and

$$(q_1p_3 - p_1q_3)^2(p_3^2 + q_3^2) + 2\left[\frac{1}{2}p_2(q_3^2 - p_3^2) - q_2q_3p_3\right]^2$$

when $a = 2b$. Hence we already know that (ω-3) and (ω-4) are integrable when $\mu := b/a = 0$ and (ω-1) is integrable when $\mu = 0, 1/2, 1$.

The Morales-Ramis theory[9] is a powerful tool to prove nonintegrability. Consider a general Hamiltonian system:

$$\dot{z} = JDH(z), \quad z \in \mathbb{C}^m \times \mathbb{C}^m, \tag{2}$$

where J is an $m \times m$ symplectic matrix

$$J = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}.$$

Let $z = \hat{z}(t)$ be a particular solution of (2). We obtain the variational equation (VE) along $z = \hat{z}(t)$

$$\dot{\xi} = JD^2H(\hat{z}(t))\eta.$$

Moreover, when the Hamiltonian system has first integrals or invariant plains, the variational equation can be reduced to a system of less linear equations called *normal variational equation* (NVE). As in introduced in the next subsection, we can define the differential Galois group G for a system of linear differential equations.

Theorem 1.1. *([9]) Let G be the differential Galois group of (NVE) of* (2) *along $z = \hat{z}(t)$. If a Hamiltonian system* (2) *is meromorphically integrable, then the identity component G^0 of G is commutative.*

Using the Morales-Ramis theory, Christov[2] stated that (ω-1), (ω-3) and (ω-4) are nonintegrable if they are not already known to be integrable as above. However his proof contained some errors. Following his approach and correcting the errors, we obtain the following theorem.

Theorem 1.2. *Let $\mu = b/a$. The following hold:*

If (ω-1) *is meromorphically integrable, then $\mu = 0, 3/10, 1/2, 3/4, 9/10$, or* 1.

If (ω-3) *is meromorphically integrable, then $\mu = 0$ or μ is written as*

$$\mu = \sqrt{\frac{k^2}{2k-1}}, \quad k \in \mathbb{Q} \text{ and } 1/2 < k \leq 1 \tag{3}$$

If (ω-4) *is meromorphically integrable, then $\mu = 0$.*

By this theorem, parameters for which (ω-4) is integrable or nonintegrable are completely determined. However, it is unknown whether or not (ω-1) for $\mu = 3/10, 3/4, 9/10$ is integrable and (ω-3) for μ written by (3) is integrable, although these systems are thought to be nonintegrable.

In Section 2, we review some of the standard facts on the differential Galois theory. In Section 3, we show the sketch of the proof of the main theorem.

2 Preliminaries

Consider a system of linear differential equations on a Riemann surface Γ

$$\dot{y} = Ay, \quad A \in \mathrm{Mat}(n, \mathcal{M}(\Gamma)), \tag{4}$$

where $\mathcal{M}(\Gamma)$ is the set of meromorphic functions on Γ. The set $\mathcal{M}(\Gamma)$ is a differential field with a derivation $\partial = \frac{d}{dt}$. We have an extension of differential fields $L \supset \mathcal{M}(\Gamma)$ called the Picard-Vessiot extension for (4) and the differential Galois group $G := \mathrm{DAut}(L/\mathcal{M}(\Gamma)) = \{\sigma \in \mathrm{Aut}(L/\mathcal{M}(\Gamma)) \mid \partial \circ \sigma = \sigma \circ \partial\}$.

If we fix a fundamental matrix Φ, then we have a faithful representation of G on the general linear group as

$$R\colon \mathrm{DAut}(L/\mathcal{M}(\Gamma)) \to \mathrm{GL}(n, \mathbb{C}), \quad \sigma \mapsto M_\sigma,$$

where $\mathrm{GL}(n, \mathbb{C})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{C}$. This representation is not unique and depends on the choice of the fundamental matrix Φ, but a different fundamental matrix only gives rise to a conjugated representation. Fixing the fundamental matrix, we can identify the image $R(G) \subset \mathrm{GL}(n, \mathbb{C})$ as the differential Galois group G. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be an algebraic group. Then it contains a unique maximal connected algebraic subgroup G^0, which is called the *connected component of the identity* or *identity component*.

Let $S \subset \Gamma$ be the set of singularities in the entries of A. We also refer to a singularity of the entries of A as that of (4). Let $t_0 \in \Gamma \setminus S$. We prolong the fundamental matrix $\Phi(t)$ analytically along any loop γ based at t_0 and containing no singular points, and obtain another fundamental matrix $\gamma * \Phi(t)$. So there exists a constant nonsingular matrix $M_{[\gamma]}$ such that

$$\gamma * \Phi(t) = \Phi(t) M_{[\gamma]}.$$

The matrix $M_{[\gamma]}$ depends on the homotopy class $[\gamma]$ of the loop γ and is called the *monodromy matrix* of $[\gamma]$.

Let $\pi_1(\Gamma \setminus S, t_0)$ be the fundamental group of homotopy classes of loops based at t_0. We have a representation

$$\tilde{R}\colon \pi_1(\Gamma \setminus S, t_0) \to \mathrm{GL}(n, \mathbb{C}), \quad [\gamma] \mapsto M_{[\gamma]}.$$

The image of $\tilde{R}$ is called the *monodromy group* of (4). As in the differential Galois group, the representation $\tilde{R}$ depends on the choice of the fundamental matrix, but the monodromy group is defined as a group of matrices up to conjugation. In general, a monodromy transformation defines an automorphism of the corresponding Picard-Vessiot extension. Hence the monodromy group is a subgroup of the differential Galois group. A singular point $t = \bar{t}$ of (4) is called *regular* if for any sector $a < \arg(t - \bar{t}) < b$ with $a < b$ there exists a fundamental matrix $\Phi(t) = (\phi_{ij}(t))$ such that for some $c > 0$ and integer N, $|\phi_{ij}(t)| < c|t - \bar{t}|^N$ as $t \to \bar{t}$ in the sector; otherwise it is called *irregular*. A system (4) is said to be *Fuchsian* if all singularities are regular. The following is useful to compute the differential Galois group of Fuchsian equations.

Theorem 2.1 (Schlessinger)**.** *Assume that a system* (4) *is Fuchsian. Then the differential Galois group of* (4) *is the Zariski closure of its monodromy group.*

Finally, we review some ways to determine whether the identity component G^0 of the differential Galois group is solvable for a second order differential equation

$$\frac{d^2x}{dz^2} + p_1(z)\frac{dx}{dz} + p_2(z)x = 0, \quad p_1(z), p_2(z) \in \mathbb{C}(z). \tag{5}$$

Using transformation $x = \exp(-\frac{1}{2}\int p_1(x)dx)y$, this equation is transformed into

$$\frac{d^2y}{dz^2} = r(z)y, \quad r(z) = -p_2(z) + \frac{1}{4}p_1(z)^2 + \frac{1}{2}\frac{dp_1}{dx}(z). \tag{6}$$

It is easy to see that G^0 of (5) is solvable if and only if that of (6) is solvable. Using Kovacic's algorithm[8], we can determine whether G^0 of (6) is solvable or not. Since we need many pages to write down the algorithm completely, we show the special case of the algorithm in order to prove nonintegrability of (ω-4).

Proposition 2.2. *Assume $r(z) \in \mathbb{C}(z)$ has only poles of order 2 at $z = a_1, \ldots, a_N, \infty$. Let $b_c = \lim_{x \to c} r(z)(z-c)^2, c = a_1, \ldots, a_N$ and $b_\infty = \lim_{z\to\infty} r(z)z^2$. If the following conditions are all satisfied, then the identity component G^0 of the differential Galois group of* (6) *is not solvable:*

(i) Let $\alpha_c^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b_c}$. For each families $s(a_1), \ldots, s(a_n), s(\infty) \in \{+, -\}$, the number

$$\alpha_\infty^{s(\infty)} - \sum_{j=1}^{N} \alpha_{a_j}^{s(a_j)}$$

is not a non-negative integer.

(ii) Let

$$E_c = \{2 + k\sqrt{1+4b_c} \mid k = 0, \pm 2\} \cup \mathbb{Z}, \quad c = a_1, \dots, a_N, \infty.$$

The all elements of $E_{a_1}, \dots, E_{a_N}, E_\infty$ *are even.*

(iii) There exists $c \in \{a_1, \dots, a_N, \infty\}$ *such that* $\sqrt{1+4b_c} \notin \mathbb{Q}$.

If (5) has only three regular singular points, Kimura's theorem [7] is more useful than Kovacic's algorithm to determine the solvability of G^0. If $z = a \in \mathbb{C}$ is a regular singular point, the solutions of the algebraic equation

$$X(X-1) + c_1 X + c_2 = 0, \quad c_j = \lim_{z \to a} (z-a)^j p_j(z).$$

are called the *characteristic exponents* of $z = a$. If $z = \infty$ is a regular singular point, the solutions of the algebraic equation

$$X(X+1) - c_1 X + c_2 = 0, \quad c_j = \lim_{z \to \infty} z^j p_j(z).$$

are called the characteristic exponents of $z = \infty$.

Theorem 2.3 (Kimura's theorem)**.** *Assume* (5) *has only three regular singular points and let* α, β, γ *be the differences of characteristic exponents at these three singular points. Then the identity component* G^0 *of the differential Galois group of* (5) *is solvable if and only if either (A) or (B) holds:*

(A) at least one of the four numbers $\alpha+\beta+\gamma$, $-\alpha+\beta+\gamma$, $\alpha-\beta+\gamma$, $\alpha+\beta-\gamma$ *is an odd;*

(B) the numbers α *or* $-\alpha$, β *or* $-\beta$ *and* γ *or* $-\gamma$ *belong (in an arbitrary order) to some of the following fifteen families:*

1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	$\mathbb{C}$	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n \in 2\mathbb{Z}$
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n \in 2\mathbb{Z}$
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n \in 2\mathbb{Z}$
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n \in 2\mathbb{Z}$
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n \in 2\mathbb{Z}$
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	$l+m+n \in 2\mathbb{Z}$
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n \in 2\mathbb{Z}$
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n \in 2\mathbb{Z}$
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	
15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n \in 2\mathbb{Z}$

Here, l, m, n are integers.

3 Sketch of the proofs

3.1 Case of $\omega = 1$

The Hamiltonian system of $(\omega\text{-}1)$ is

$$\begin{aligned} \dot{q}_1 &= 2a(p_1p_2 + q_1q_2), & \dot{p}_1 &= 2a(q_1p_2 - p_1q_2), \\ \dot{q}_2 &= a(p_1^2 - q_1^2) + b(p_3^2 - q_3^2), & \dot{p}_2 &= -2ap_1q_1 - 2bp_3q_3, \\ \dot{q}_3 &= 2b(p_2p_3 + q_2q_3), & \dot{p}_3 &= 2b(p_2q_3 - q_2p_3). \end{aligned}$$

This has a particular solution of the form

$$\begin{aligned} q_1(t) &= -\sqrt{2}iq_2(t), \quad & q_2(t) &= \frac{1}{2}\frac{\wp'(2at)}{\wp(2at)}, \quad & q_3(t) &= 0, \\ p_1(t) &= \sqrt{2}ip_2(t), \quad & p_2(t) &= -\frac{C}{3}\frac{1}{\wp(2at)}, \quad & p_3(t) &= 0, \end{aligned}$$

where $\wp$ is Weierstrass's $\wp$ function

$$\wp'^2 = 4\wp^3 - g_3, \quad g_3 = -\frac{4C^2}{27}, \quad C = -\frac{H^2(q(t), p(t))}{2a}.$$

The normal variational equation for $q_3 = p_3 = 0$ is

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}2bq_2(t) & 2bp_2(t)\\ 2bp_2(t) & -2bq_2(t)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}. \tag{7}$$

By the transformation $z = \wp'(2at)$, we get

$$\frac{d}{dz}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{\mu z}{3(z^2+g_3)} & -\frac{2C\mu}{9(z^2+g_3)}\\ -\frac{2C\mu}{9(z^2+g_3)} & -\frac{\mu z}{3(z^2+g_3)}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

where $\mu = b/a$. We rewrite the equation as a second order equation

$$\frac{d^2\xi}{dz^2} + \frac{54z}{27z^2 - 4C^2}\frac{d\xi}{dz} - \frac{9\mu(4C^2(\mu-1) + 9(\mu+3)z^2)}{(27z^2 - 4C^2)^2} = 0,$$

which has three regular singular points at $z = \pm\frac{2\sqrt{3}}{9}C, \infty$. The characteristic exponents of $z = \pm\frac{2\sqrt{3}}{9}C$ are $\frac{1}{3}\mu, -\frac{1}{3}\mu$ and those of $z = \infty$ are $\frac{1}{3}\mu, 1 - \frac{1}{3}\mu$. Thus the differences of the characteristic exponents are $\alpha = \frac{2}{3}\mu, \beta = \frac{2}{3}\mu$ and $\gamma = 1 - \frac{2}{3}\mu$.

Proposition 3.1. *Assume $\mu \neq 0$, the differential Galois group of* (7) *is solvable if and only if $\mu = 3/10, 1/2, 3/4, 9/10, 1$.*

Proof. We use Kimura's theorem to prove the proposition. Since $0 < \mu \leq 1$, $\alpha + \beta + \gamma = 1 + \frac{2}{3}\mu$ and $-\alpha + \beta + \gamma = 2\mu/3$ can not be odd. If $\alpha + \beta - \gamma = 2\mu - 1$ is odd, $\mu = 1$. Hence this equation falls into (A) only when $\mu = 1$.

Since the difference of the exponents have same denominator, this equation does not fall into cases of $2, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15$. If this equation falls into case 1, then $2\mu/3 = 1/2$, i.e., $\mu = 3/4$. If this equation falls into case 3, then $2\mu/3 = 1/3$, i.e., $\mu = 1/2$. If this equation falls into case 11, then $-2\mu/3 = 2/5 - 1$, i.e., $\mu = 9/10$. If this equation falls into case 13, then $1 - 2\mu/3 = 4/5$, i.e., $\mu = 3/10$. Hence this equation falls into (B) only when $\mu = 3/10, 1/2, 3/4, 9/10$.

□

Remark 3.2. *Equation* (7) *is Fuchsian equations on a torus. Generally, it is difficult to compute monodromy matrices of equations on a torus. Christov[2] computed local monodromy matrices and stated that G^0 is not commutative. However, non-commutativity of the local monodromy matrices does not mean non-commutativity of the monodromy group M. Moreover, G^0 may be commutative even if M is not commutative.*

3.2 Case of (ω-3)

The Hamiltonian system for Hamiltonian (ω-3) is

$$\begin{aligned}
\dot{q}_1 &= 2a(p_1p_2 + q_1q_2) + b(p_2p_3 + q_2q_3), & \dot{p}_1 &= 2a(p_2q_1 - p_1q_2) + b(q_2p_3 - p_2q_3),\\
\dot{q}_2 &= a(p_1^2 - q_1^2) + b(p_1p_3 + q_1q_3), & \dot{p}_2 &= -2ap_1q_1 + b(q_1p_3 - p_1q_3),\\
\dot{q}_3 &= b(p_1p_2 - q_1q_2), & \dot{p}_3 &= -b(q_1p_2 + p_1q_2).
\end{aligned}$$

This has a particular solution of the form

$$\begin{aligned}
q_1(t) &= \tfrac{\sqrt{2}sF}{\sqrt{2s^2+3}}\tfrac{1}{\cosh(bsFt)}, & q_2(t) &= \tfrac{F}{\sqrt{2}}\tanh(bsFt), & q_3(t) &= \tfrac{F}{\sqrt{2s^2+3}}\tfrac{1}{\cosh(bsFt)}\\
p_1(t) &= 0, & p_2(t) &= 0, & p_3(t) &= 0,
\end{aligned}$$

where $s = -\frac{\mu+\sqrt{\mu^2-1}}{\sqrt{2}}$ and F is a number such that $F = H^2(q(t), p(t))$.

Using the invariant plain $p_1 = p_2 = p_3 = 0$ and the first integral H^2, we obtain (NVE)

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}\frac{b}{3}(6\mu - \sqrt{2}s)q_2(t) & \frac{b}{3}\frac{(3-4s^2)q_3(t)^2-F^2}{q_3(t)}\\ \frac{2b}{3}(2s^2+3)q_3(t) & -\frac{2\sqrt{2}bs}{3}q_2(t)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}.$$

By the transformation $x = \frac{1}{2}(\tanh(bFst) + 1)$, we can reduce the NVE to the hypergeometric equation

$$\frac{d^2\xi_2}{dx^2} - k\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{d\xi}{dx} + \left[k\left(\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + 2\frac{k-1}{x(x-1)}\right]\xi_2 = 0,$$

where $k = \frac{\mu}{\mu+\sqrt{\mu^2-1}}$. Note that k is not a real number for $0 < \mu < 1$ and $1/2 < k \leq 1$ for $1 \leq \mu$. The characteristic exponents of $x = 0, 1$ are $1, k$ and those of $x = \infty$ are $-2k+1, -2$. Since this hypergeometric equation is reducible, the identity component of G is always solvable. Hence we can't use to Kovacic's algorithm and Kimura's theorem in order to prove nonintegrability of (ω-3).

We compute the differential Galois group directly by using the monodromy matrices. When $k \neq 1$, we obtain

$$M_0 = \begin{pmatrix}1 & 0\\ 0 & \kappa\end{pmatrix},\ M_1 = \begin{pmatrix}1 & 1\\ 0 & \kappa\end{pmatrix},\ \kappa = e^{-2\pi ik}$$

from the formula for monodromy matrices of a reducible hypergeometric equation [6].

Lemma 3.3. *Let G be the differential Galois group of (NVE). If $k \in \mathbb{Q}$, then G^0 is commutative. If $k \notin \mathbb{Q}$, then G^0 is not commutative.*

Proof. Let $M = \langle M_0, M_1\rangle$ be the monodromy group. By the Schlessinger theorem, we obtain $G = \bar{M}$.

If $k \in \mathbb{Q}$, then there is $N \in \mathbb{N}$ such that $\kappa^N = 1$. In this case,

$$G = \bar{M} \subset \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}, b^N = 1 \right\}.$$

Hence,

$$G^0 \subset \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{C} \right\}$$

and G^0 is commutative.

If k is not a rational number, then κ is not a root of the unity. Hence

$$\overline{\langle M_1 \rangle} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C}^* \right\}$$

and thus

$$G = \bar{M} \supset \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C}^* \right\}.$$

Noting that $G^0 = (\bar{M})^0$ is a normal subgroup of G and $M_1 \in G^0, M_2 \in G$, we have

$$M_2^{-1} M_1 M_2 = \begin{pmatrix} 1 & 1-\kappa \\ 0 & \kappa \end{pmatrix} \in G^0.$$

The matrix $\begin{pmatrix} 1 & 1-\kappa \\ 0 & \kappa \end{pmatrix}$ is not commutative with M_1 and then G^0 is not commutative. $\square$

3.3 Case of (ω-4)

The Hamiltonian system for Hamiltonian (ω-4) is

$$\begin{aligned} \dot{q}_1 &= 2a(q_1 q_2 + p_1 p_2), & \dot{p}_1 &= 2a(p_2 q_1 - p_1 q_2), \\ \dot{q}_2 &= a(p_1^2 - q_1^2) + 2b(p_2 p_3 + q_2 q_3), & \dot{p}_2 &= -2a p_1 q_1 + 2b(q_2 p_3 - p_2 q_3), \\ \dot{q}_3 &= b(p_2^2 - q_2^2), & \dot{p}_3 &= -2b p_2 q_2. \end{aligned}$$

This has a particular solution of the form

$$\begin{aligned} q_1(t) &= 0, \quad q_2(t) = \tfrac{C}{3} \tfrac{1}{\wp(ib\sqrt{2}t)}, \quad q_3(t) = \tfrac{i}{\sqrt{2}} p_2(t), \\ p_1(t) &= 0, \quad p_2(t) = -\tfrac{1}{2} \tfrac{\wp'(ib\sqrt{2}t)}{\wp(ib\sqrt{2}t)}, \quad p_3(t) = \tfrac{i}{\sqrt{2}} q_2(t), \end{aligned}$$

where $\wp$ is Weierstrass's elliptic function, which satisfies

$$\wp'^2 = 4\wp^3 - g_3, \quad g_3 = -\frac{4C^2}{27}, \quad C = -\frac{i\sqrt{2} H_2(q(t), p(t))}{b} \neq 0.$$

The normal variational equation is written as

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} -i\sqrt{2}\mu\frac{C}{3\wp(ib\sqrt{2}t)} & i\sqrt{2}\mu\frac{\wp'(ib\sqrt{2}t)}{2\wp(ib\sqrt{2}t)} \\ i\sqrt{2}\mu\frac{\wp'(ib\sqrt{2}t)}{2\wp(ib\sqrt{2}t)} & i\sqrt{2}\mu\frac{C}{3\wp(ib\sqrt{2}t)}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

By the transformation $z = \wp'(ib\sqrt{2}t)$, the equation is transformed into

$$\frac{d^2\xi}{dz^2} - \frac{27z^2+4C^2}{z(27z^2-4C^2)}\frac{d\xi}{dz} - \frac{6(27\mu z^3 - 27\sqrt{2}iCz^2 + 12\mu C^2 z + 4\sqrt{2}iC^3)}{z(27z^2-4C^2)^2}\xi = 0. \tag{8}$$

To determine wether G^0 of this equation is solvable or not, we use Kovacic's algorithm. By the transformation $\xi = \exp(\frac{1}{2}\int \frac{27z^2+4C^2}{z(27z^2-4C^2)})\eta$, this equation become

$$\frac{d^2\eta}{dz^2} = r(z)\eta,$$
$$r(z) = -\frac{3}{4}\frac{(243+216\mu^2)z^4 - 216\sqrt{2}i\mu Cz^3 + (216+96\mu^2)C^2z^2 + 32\sqrt{2}iC^3\mu - 16C^4}{z^2(27z^2-4C^2)^2}.$$

Proposition 3.4. *For $\mu \neq 0$, the identity component G^0 of the differential Galois group of* (8) *is not solvable.*

Proof. Poles of $r(z)$ are $x = 0, \pm\frac{2\sqrt{3}C}{9}, \infty$ and the those orders are all 2. Let $a_\pm = \pm\frac{2\sqrt{3}C}{9}$. The coefficients are

$$b_0 = \frac{3}{4}, \quad b_{a_+} = b_{a_-} = b_\infty = -\frac{2}{9}\mu^2 - \frac{1}{4}.$$

We obtain $\alpha_0^\pm = \frac{1}{2} \pm 1, \alpha_{a_+}^\pm = \alpha_{a_-}^\pm = \alpha_\infty^\pm = \frac{1}{2} \pm \frac{2\sqrt{2}}{3}\mu i$. Hence the imaginary part of

$$\alpha_\infty^{s(\infty)} - \alpha_0^{s(0)} - \alpha_{a_+}^{s(a_+)} - \alpha_{a_-}^{s(a_-)}$$

is non-zero, and condition (i) of Proposition 2.2 holds.

We have $E_0 = \{-2, 2, 6\}$, $E_c = \{2\}$. Hence condition (ii) holds.

Since $\sqrt{1+4b_{a_+}} = \frac{2\sqrt{2}}{3}\mu i \notin \mathbb{Q}$, condition (iii) holds. □

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References

[1] E. van der Aa, First order resonances in three degrees of freedom systems Celestial Mech. Dynam. Astronom. 31 (1983), 163–191.

[2] O. Christov, Non-integrability of first order resonances in Hamiltonian systems in three degrees of freedom, Celestial Mech. Dynam. Astronom. 112 (2012), 149–167.

[3] J. Duistermaat, Non-integrability of the 1:1:2 resonance, Ergod. Theory Dyn. Syst. 4, (1984), 553–568.

[4] H. Ito, Convergence of Birkhoff normal forms for integrable systems, Comment. Math. Helv. 64 (1989), 412–461.

[5] H. Ito, Integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case, Math. Ann. 292 (1992), 411–444.

[6] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé: A Modern Theory of Special Functions, Friedr. Vieweg & Sohn, Braunschweig, 1991.

[7] T. Kimura, On Riemann's equations which are solvable by quadratures, Funkcial. Ekvac. 12 (1969) 269–281.

[8] J.J. Kovacic, An algorithm for solving second order linear homogeneous differential equations, J. Symbolic Comput. 2 (1986), 3–43.

[9] J.J. Morales-Ruiz, J. Ramis, Galoisian obstructions to integrability of Hamiltonian systems I. Methods Appl. Anal. 8 (2001), 33-95.

[10] J.M. Tuwankotta, F. Verhulst, Symmetry and integrability in Hamiltonian normal forms, SIAM J. Appl. Math 61 (2000), 1369–1385.

[11] N.T. Zung, Convergence versus integrability in Birkhoff normal form, Ann. of Math. 161 (2005), 141–156.

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